Answer Key

PHYSICS

1. b	2. c	3. a	4. c	5. a	6. a	7. c	8. d	9. d	10. c
11. c	12. b	13. c	14. c	15. b	16. d	17. a	18. c	19. c	20. b
21. 336	22. 0	23. 2	24. 12.6	25. 162					

CHEMISTRY

1. d	2. a	3. b	4. d	5. b	6. b	7. d	8. a	9. b	10. a
11. a	12. b	13. a	14. a	15. c	16. a	17. d	18. d	19. b	20. a
21. 3	22. 4	23. 11	24. 8	25. 149					

MATHEMATICS

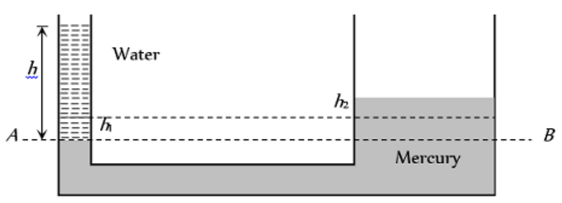
1. a	2. b	3. b	4. b	5. c	6. d	7. a	8. c	9. b	10. d
11. c	12. a	13. a	14. d	15. c	16. a	17. a	18. c	19. c	20. c
21. 4	22. 6	23. 3	24. 99	25. 8					

Question-wise Detailed Solution

PHYSICS

1. Two communicating vessels contain mercury. The diameter of one vessel is n times larger than the diameter of the other. A column of water of height h is poured into the left vessel. The mercury level will rise in the right-hand vessel (s = relative density of mercury and $\rho =$ density of water) by

Solution



If the level in narrow tube goes down by h_1 then in wider tube goes up to h_2,

Now, $\pi r^2 h_1 = \pi (nr)^2 h_2$

$\Rightarrow h_1 = n^2 h_2$

Now, pressure at point A $=$ pressure at point B

$h\rho g = (h_1 + h_2)\rho' g$

$\Rightarrow h = (n^2 h_2 + h_2) s g \left(\text{As } s = \frac{\rho'}{\rho}\right)$

$\Rightarrow h_2 = \frac{h}{(n^2+1)s}$

2 A uniformly wound solenoid coil of self-inductance 1.8×10^{-4} H and resistance $6\ \Omega$ is broken up into two identical coils. These identical coils are then connected in parallel across a 12 V battery of negligible resistance. The time constant for the current in the circuit is

Solution

Given, self-inductance, $L = 1.8 \times 10^{-4}\ \mathrm{H}$

Resistance, $R = 6\ \Omega$

When self-inductance and resistance are broken up into identical coils.

Then, the self-inductance of each oil

$= \frac{1.8\times10^{-4}}{2}\ \mathrm{H}$

The resistance of each oil

$= \frac{6\Omega}{2} = 3\ \Omega$

Coil are then connected in parallel
$(L_{eq} = \frac{L_1 L_2}{L_1 + L_2})$

$\therefore\ L' = \frac{\frac{1.8}{2}\times10^{-4}\times\frac{1.8}{2}\times10^{-4}}{\frac{1.8}{2}\times10^{-4}+\frac{1.8}{2}\times10^{-4}}$

$= 0.45 \times 10^{-4}\ \mathrm{H}$

and $R' = \frac{3\times3}{3+3} = 1.5\Omega$
$(R_{eq} = \frac{R_1 R_2}{R_1 + R_2})$

Time constant $= \frac{L'}{R\prime}$

$= \frac{0.45\times10^{-4}}{1.5} = 0.3 \times 10^{-4}\ \mathrm{s}$

3 A large temple has a depression in one wall. On the floor plan, it appears as an indentation having a spherical shape of radius $2.50\ \mathrm{m}$. A worshiper stands on the centerline of the depression, $2.00\ \mathrm{m}$ out from its deepest point, and whispers a prayer. Where is the sound concentrated after reflection from the black wall of the depression?

Solution

Wall will act as concave mirror

$\frac{1}{\mathrm{v}} + \frac{1}{\mathrm{u}} = \frac{1}{\mathrm{f}} \quad \Rightarrow \quad \frac{1}{\mathrm{v}} + \frac{1}{\mathrm{u}} = \frac{2}{\mathrm{R}} \quad \Rightarrow \quad \frac{1}{\mathrm{v}} + \frac{1}{-2} = \frac{2}{-2.5}$

$\frac{1}{\mathrm{v}} = \frac{2}{-2.5} + \frac{1}{2} \quad \Rightarrow \quad \frac{1}{\mathrm{v}} = \frac{-8+5}{10}$

$\frac{1}{\mathrm{v}} = \frac{-3}{10}$

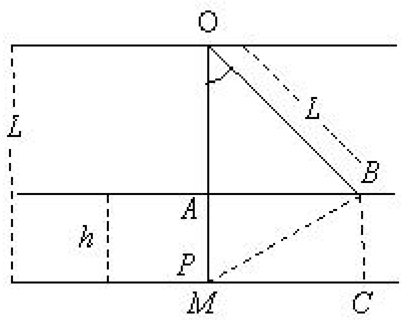
$\mathrm{v} = \frac{-10}{3}\mathrm{m}$

4 If a simple pendulum of length, L has maximum angular displacement α, then the maximum kinetic energy of bob of mass M is

Solution

Given, mass of bob=M

Length of simple pendulum=L



Now, in $\Delta OAB, \quad \cos\alpha = \frac{OA}{OB}$

or $\cos\alpha = \frac{OA}{L}$

or $AO = L\cos\alpha$

and $AP = OP - AO$

$AP = L - L\cos\alpha$

or $h = L(1 - \cos\alpha)$

The maximum kinetic energy of bob

= Maximum potential energy

$\mathrm{KE} = Mgh$

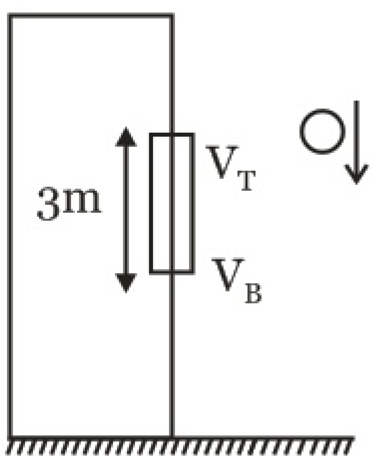
$\mathrm{KE} = MgL(1 - \cos\alpha)$

5 A ball is dropped from the top of a building. The ball takes 0.5 s to fall past the 3 m length of a window some distance from the top of the building. If the speed of the ball at the top and at the bottom of the window are v_T and v_B respectively, then (take g = 10 m/s^2) :

Solution

$\mathrm{s} = \frac{(\mathrm{u}+\mathrm{v})}{2}\mathrm{t}$

$3 = \frac{(\mathrm{v}_\mathrm{T}+\mathrm{v}_B)}{2} \times 0.5$



$\mathrm{v}_\mathrm{T} + \mathrm{v}_B = 12\mathrm{m/s}$

Also $\mathrm{v}_\mathrm{B} = \mathrm{v}_\mathrm{T} + (9.8)(0.5)$

$v_B - v_T = 4.9 m/s$

6 The kinetic energy K of a particle moving in a straight line depends on the distance s as K = as². The force acting on the particle is

(where a is a constant)

Solution

Here kinetic energy $K = \frac{1}{2}mv^2 = a\,s^2$

∴ $mv^2 = 2as^2$ ...(i)

Differentiating eqn (i) w.r.t. to time t, we get

$$2mv\frac{dv}{dt} = 4as\frac{ds}{dt} = 4\,asv \quad \text{or} \quad m\frac{dv}{dt} = 2as$$

7 A sample of ^{18}F is used internally as a medical diagnostic tool to look for the effects of the positron decay $\left(T_{\frac{1}{2}} = 110 \text{ min}\right)$. How long does it take for 99% of the ^{18}F to decay?

Solution

Radioactive decay equation is

$$N = N_0 e^{-\lambda t} = N_0 e^{-\ln(2)\frac{t}{T_{\frac{1}{2}}}}$$

After decay of 99% of the initial sample only 1% will be left, and $\frac{N}{N_0} = 1\%$. We then have

$$\frac{N}{N_0} = \frac{1}{100} = e^{-\ln(2)\frac{t}{T_{\frac{1}{2}}}}$$

If we take the natural logarithm, we have

$$-\ln 100 = -\frac{\ln 2 \times t}{T_{\frac{1}{2}}}$$

Which on solving for t yields

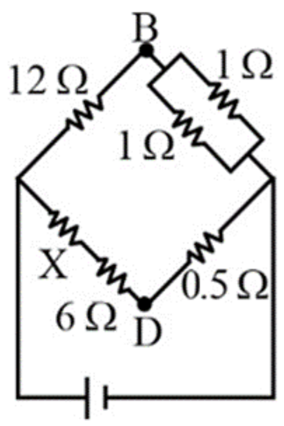
$$\therefore t = \frac{\ln 100}{\ln 2} \times T_{\frac{1}{2}}$$

$$= \frac{\log 100}{\log 2} \times T_{\frac{1}{2}}$$

$$= \frac{2}{0.3010} \times 110$$

$$= 731 \text{ min} = 12.2\ h$$

8 The value of unknown resistance X for which the potential difference between B and D will be zero in arrangement of given figure is



Solution

According to balanced Wheatstone bridge

$\frac{12}{X+6} = \frac{1/2}{0.5} \Rightarrow X + 6 = 12 \Rightarrow X = 6\,\Omega$

9 A bus is moving with a speed of 10 m s^{-1} on a straight road. A scooterist wishes to overtake the bus in 100 s. If the bus is at a distance of 100 m from the scooterist with what speed should the scooterist chase the bus ? (Assume that the scooterist move with constant speed)

Solution

Let v_s be the velocity of the scooter, the distance between the scooter and the bus = 100 m,

The velocity of the bus = 10 m s^{-1}

Time taken to overtake = 100 s

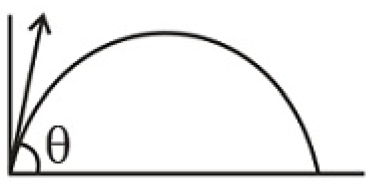
Relative velocity of the scooter with respect to the bus $= v_s - 10$

$\therefore t = \frac{100}{(v_s - 10)} = 100\ s \Rightarrow v_s = 11\ m\ s^{-1}$

10 The trajectory of a projectile in vertical plane in $y = ax - bx^2$, where a and b are constant and x and y are respectively horizontal and vertical distances of the projectile from the point of projection. The maximum height attained by the particle and the angle of projection from the horizontal is:

Solution

Let the angle of projection be θ



$y = ax - bx^2$

To get maxima, $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = a - b.2x = 0$

$\Rightarrow x = \frac{a}{2b}$

$y_{max} = \frac{a^2}{2b} - \frac{ba^2}{4b^2} = \frac{a^2}{4b}$

Angle of projections can be obtained by slope $\left(\frac{dy}{dx}\right)$ at $x = 0$

$\Rightarrow$ Slope at $x = 0$ is a

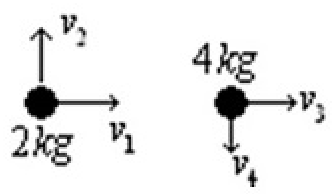
$\Rightarrow \tan\theta = a$

$\Rightarrow \theta = \tan^{-1}(a)$

11 A particle of mass $2\ kg$ moving with a speed of $6\ m/s$ collides elastically with another particle of mass $4\ kg$ traveling in same direction with a speed of $2\ m/s$. The maximum possible deflection of the $2\ kg$ particle is

Solution

Let situations after collision for balls are



According to question $2v_1 + 4v_3 = 20$

$2v_2 = 4v_4$

$\frac{1}{2} \times 2\left(v_1^2 + v_2^2\right) + \frac{1}{2} \times 4\left(v_3^2 + v_4^2\right) = \frac{1}{2} \times 2 \times (6)^2 + \frac{1}{2} \times 4 \times (2)^2 = 44$

i.e. $v_1^2 + v_2^2 + 2v_3^2 + \frac{v_2^2}{2} = 44$

i.e. $v_1^2 + \frac{3}{2}v_2^2 + \left(\frac{10-v_1}{2}\right)^2 2 = 44$

i.e. $2v_1^2 + 3v_2^2 + 100 - 20v_1 + v_1^2 = 88$

i.e. $3v_1^2 + 3v_2^2 - 20v_1 + 12 = 0$

i.e. $\frac{12}{v_1^2} - \frac{20}{v_1} + \frac{3v_2^2}{v_1^2} + 3 = 0$

Quadratic in $\frac{1}{v_1}$ which is real

$\therefore D \geq 0 \ \therefore (-20)^2 - 4 \times 12\left(1 + \frac{v_2^2}{v_1^2}\right) \times 3 \geq 0$

$\therefore \ \ 1 + \frac{v_2^2}{v_1^2} \leq \frac{400}{144}$

$\frac{v_2^2}{v_1^2} \leq \frac{256}{144}$

$-\frac{16}{12} \leq \frac{v_2}{v_1} \leq \frac{16}{12} \Rightarrow k = \tan\theta \leq \frac{4}{3} \Rightarrow \theta \leq 53^o$

12 A small satellite of mass m is revolving around the earth in a circular orbit of radius r_0 with speed v_0. At a certain point in its orbit, the direction of motion of satellite is suddenly changed by angle $\theta = \cos^{-1}(3/5)$ by turning its velocity vector in the same plane of motion, such that the speed remains constant. The satellite consequently goes into an elliptical orbit around earth. The ratio of speed at perigee to speed at apogee is

Solution

Using conservation of angular momentum about O.

$\mathrm{mV_P r_P = mV_A r_A = mV_0 r_0 cos}\theta$

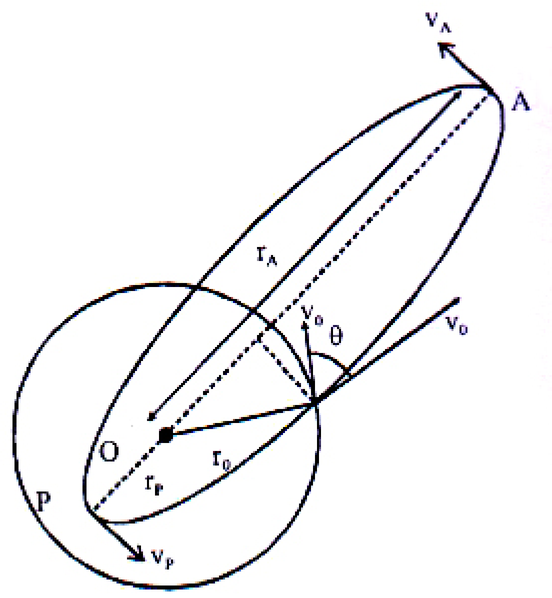
$\mathrm{V_A r_A = V_P r_P} = \frac{3\mathrm{v_0 r_0}}{5}$ ......(1)

using conservation of energy

$\frac{1}{2}\mathrm{mV_A^2} + \frac{-\mathrm{GMm}}{\mathrm{r_A}} = \frac{-\mathrm{GMm}}{\mathrm{r_0}} + \frac{1}{2}\mathrm{mV_0^2}$

$\Rightarrow \frac{9\mathrm{V_0^2 r_0^2}}{50\mathrm{r_A^2}} - \frac{\mathrm{V_0^2 r_0}}{\mathrm{r_A}} + \frac{\mathrm{V_0^2}}{2} = 0$ $\left[\mathrm{Let} \frac{\mathrm{r_0}}{\mathrm{r_A}} = \mathrm{x}\right]$ This is a quadratic equation in $\left(\frac{r_0}{r_a}\right)$

$\Rightarrow 9x^2 - 50x + 25 = 0 \Rightarrow x = 5 \ or \ x = \frac{5}{9}$



$\Rightarrow r_A = \frac{9}{5} r_o \ ....(1)$

$r_p = \frac{r_o}{5} \quad ....\left(2\right)$

using (1)&(2)

$\Rightarrow \frac{V_P}{V_A} = \frac{r_A}{r_P} = 9$

13 In C.G.S. system, the magnitude of the force is 100 dynes. In another system where the fundamental physical quantities are kilogram, metre and minute, the magnitude of the force is

Solution

$n_2 = n_1 \left(\frac{M_1}{M_2}\right)^1 \left(\frac{L_1}{L_2}\right)^1 \left(\frac{T_1}{T_2}\right)^{-2}$

$= 100 \left(\frac{\text{gm}}{\text{kg}}\right)^1 \left(\frac{\text{cm}}{\text{m}}\right)^1 \left(\frac{\text{sec}}{\text{min}}\right)^{-2}$

$= 100 \left(\frac{\text{gm}}{10^3 \text{ gm}}\right)^1 \left(\frac{\text{cm}}{10^2 \text{ cm}}\right)^1 \left(\frac{\text{sec}}{60 \text{ sec}}\right)^{-2}$

$n = \frac{3600}{10^3} = 3.6$

14 An ideal gas is expanding such that pT^2= constant. The coefficient of volume expansion of the gas is

Solution

Given, pT^2 = constant

$\therefore \quad \left(\frac{nRT}{V}\right) T^2 =$constant

or $T^3 V^{-1} =$ constant

Differentiating the equation, we get

$\frac{3T^2}{V} . dT - \frac{T^3}{V^2} . dV = 0$

or $3. dT = \frac{T}{V} . dV$

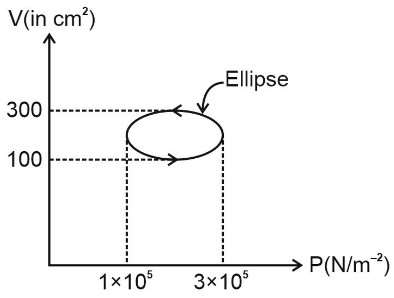
From the equation, $dV = V\gamma dT$

$\gamma =$ coefficient of volume expansion of gas

$= \frac{dV}{V.dT}$

$\therefore \qquad \gamma = \frac{dV}{V.dT} = \frac{3}{T}$

15 Calculate the heat absorbed by a system in going through the cyclic process shown in diagram.



Solution

The cycle i clockwise; thus net work done by the gas. As in a cyclic process no change in internal energy takes place, heat supplied is equal to the work done by the gas in one complete cycle. So in this case heat supplied to the gas is given as

Q = work done by the gas, W

= area of ellipse

$= \pi \mathrm{ab}$

where a and b are semi-major and semi-minor axes of the ellipse, respectively, which are given from the diagram as

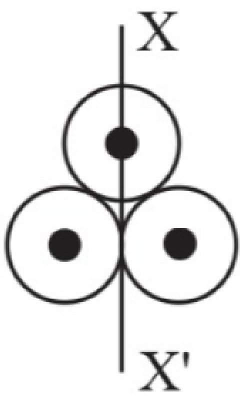
a = 1.0 x 10^5 N/m and b = 100 x 10^{-6} m^3

Thus area of ellipse is

$$\mathrm{Q} = \mathrm{W} = \pi \times 1.0 \times 10^5 \times 100 \times 10^{-6}$$

$$= 3.14 \times 10 = 31.4 \mathrm{~J}$$

16 Three identical spherical shells, each of mass m and radius r are placed as shown in the figure. Consider an axis $\mathrm{XX}^{\prime}$ which is touching the two shells and passing through diameter of the third shell. Moment of inertia of the system consisting of these three spherical shells about $\mathrm{XX}^{\prime}$ axis is:



Solution

Total MI of the system,

$\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3$

$\mathrm{I}_2 = \mathrm{I}_3 = \frac{2}{3} \mathrm{mr}^2 + \mathrm{mr}^2 = \frac{5 \mathrm{mr}^2}{3}$

$\mathrm{I}_1 = \frac{2}{3} \mathrm{mr}^2$

$\therefore \mathrm{I} = 2 \times 5 \frac{\mathrm{mr}^2}{3} + \frac{2}{3} \mathrm{mr}^2$

$= \frac{12 \mathrm{mr}^2}{3} = 4 \mathrm{~mr}^2$

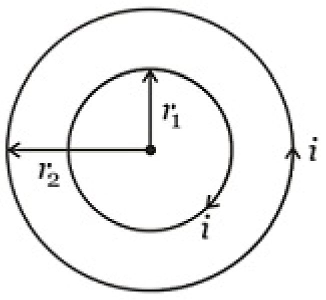
17 A particle of mass m rotates in a circle of the radius a with a uniform angular speed ω_0 . It is viewed from a frame rotating about the z-axis with a uniform angular speed ω. The centrifugal force on the particle is

Solution

Centrifugal force does not depend upon angular velocity of particle. It depends upon angular velocity of frame of reference.

$\therefore$ Centrifugal force is $\mathrm{F} = \mathrm{m}\omega^2\mathrm{a}$

18 Two circular concentric loops of radii $\mathrm{r}_1 = 20\ \mathrm{cm}, \mathrm{r}_2 = 30\ \mathrm{cm}$ are placed in the $\mathrm{X} - \mathrm{Y}$ plane as shown in the figure. A current $\mathrm{I} = 7\ \mathrm{A}$ is flowing through them. The magnetic moment of this loop system is



Solution

Here, magnetic moment due to loop 1

$\mathrm{M}_1 = \mathrm{iA}_1$

$= \mathrm{i}\pi\mathrm{r}_1^2$

$= 7\pi(0\ .20)^2 = 0\ .28\ \pi$

Similarly magnetic moment due to loop-2

$\mathrm{M}_2 = \mathrm{iA}_2 = \mathrm{i}\pi\mathrm{r}_2^2$

$= 7\pi\ (0\ .30)^2 = 0\ .63\ \pi$

Net magnetic moment

$= \mathrm{M}_1 - \mathrm{M}_2$

$= 0\ .63\ \pi - 0\ .28\ \pi = 1\ .1\ \ \mathrm{A\ m}^2$

19 Monochromatic radiation of wavelength λ is incident on a hydrogen sample in ground state. Hydrogen atoms absorb a fraction of light and subsequently emit radiation of six different wavelengths. Find the value of λ.

Solution

As the hydrogen atoms emit radiation of six different wavelengths, some of them must have been excited to $\mathrm{n} = 4$, The energy in $\mathrm{n} = 4$ state is

$\mathrm{E}_4 = \frac{\mathrm{E}_1}{4^2} = -\frac{13\cdot 6\mathrm{eV}}{16} = -0\cdot 85\mathrm{eV}.$

The energy needed to take a hydrogen atom from its ground state to $\mathrm{n} = 4$ is

13.6 eV - 0.85 eV $= 12.75$ eV .

The photons of the incident radiation should have $12.75\ \mathrm{eV}$ of energy. So

$\frac{\mathrm{hc}}{\lambda} = 12\cdot 75\mathrm{eV}$

or , $\lambda = \frac{\mathrm{hc}}{12\cdot 75\mathrm{eV}}$

$= \frac{1242\mathrm{eV\ \ nm}}{12\cdot 75\mathrm{eV}}$

$= 97.5\ \mathrm{nm}.$

20 The pitch of a screw gauge is 0.5 mm and there are 50 divisions on the circular scale. In measuring the thickness of a metal plate, there are five divisions on the pitch scale (or main scale) and thirty fourth division coincides with the reference line. Calculate the thickness of the metal plate

Solution

Pitch, p = 0.5 mm

Number of circular scale divisions = 50

∴ Least count, $\mathrm{C}=\frac{0.5}{50}=0.01 \mathrm{~mm}$

Main scale reading = 5 (pitch)

= 5 (0.5) = 2.5 mm

Circular scale reading = NC = 34 (0.01)

= 0.34 mm

∴ Total reading = 2.5 + 0.34 = 2.84 mm

21 In a resonance tube experiment, to determine the speed of sound in air, a pipe of diameter $5 \mathrm{~cm}$ is used. The air column in the pipe resonates with a tuning fork of frequency $480 \mathrm{~Hz}$ when the minimum length of the air column is $16 \mathrm{~cm}$. Find the speed of sound (in $\mathrm{m} \mathrm{s}^{-1}$) in the air at room temperature.

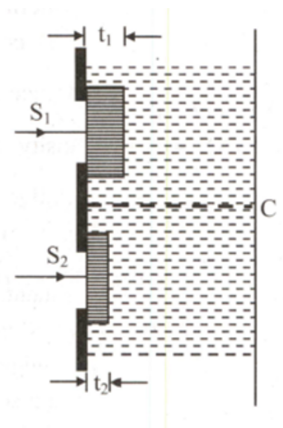
Solution

$$\frac{v}{4[L+0.6r]}=480$$

$$\Rightarrow v = 480 \times 4 \times [L + 0.6r]$$

$$\Rightarrow v=336 \mathrm{~m} \mathrm{~s}^{-1}$$

22 A screen is at a distance $\mathrm{D}=80\mathrm{cm}$ from a diaphragm having two narrow slits S_1 and S_2 which are $\mathrm{d}=2\mathrm{mm}$ apart. Slit S_1 is covered by a transparent sheet of thickness $\mathrm{t}_1=2.5\mu\mathrm{m}$ and slit S_2 is covered by another sheet of thickness $\mathrm{t}_2=1.25\mu\mathrm{m}$. (as in figure). Both sheets are made of same material having refractive index $\mu_{\mathrm{s}}=1.40$. Water is filled in the space between diaphragm and screen. A monochromatic light beam of wavelength $\lambda=5000\text{Å}$ is incident normally on the diaphragm. Assuming intensity of beam to be uniform, calculate the ratio of intensity of C to maximum intensity of interference pattern obtained on the screen $\left(\mu_{\mathrm{w}}=\frac{4}{3}\right)$



Solution

Path difference at C,

$$\Delta \mathrm{x}=\mathrm{t}_1(\mu-1)-\mathrm{t}_2(\mu-1)$$
$$=(\mathrm{t}_1-\mathrm{t}_2)(\mu-1)$$
$$=(\mathrm{t}_1-\mathrm{t}_2)\left(\frac{\mu_{\mathrm{s}}}{\mu_{\mathrm{w}}}-1\right)$$
$$=(\mathrm{t}_1-\mathrm{t}_2)\left(\frac{\mu_{\mathrm{s}}}{\mu_{\mathrm{w}}}-1\right)$$
$$=(2.5-1.25)\left(\frac{14\times 3}{4\times 10}-1\right)\mu\mathrm{m}$$
$$=\frac{25}{400}\mu\mathrm{m}$$

$= \frac{1}{16} \mu\text{m}$
$\Rightarrow \phi = \frac{2\pi}{\lambda} \times \Delta\text{x}$
$\frac{2\pi \times 4}{5000 \times 10^{-10} \times 3} \times \frac{1}{16} \times 10^{-6} \cdots \left(\because \lambda' = \frac{\lambda}{\mu}\right)$
$= \frac{\pi}{3}$
$\Rightarrow \text{I}_{\text{max}} = 4\text{I}_0$
Intensity at $\text{C}, \text{I}_\text{c} = 2\text{I}_0 \left(1 + \cos\frac{\pi}{3}\right) = 3\text{I}_0$
$\therefore$ Required ratio $= \frac{\text{I}_\text{c}}{\text{I}_{\text{max}}} = \frac{3}{4} = 0.75$

23 A solid sphere of radius R has a charge Q distributed in its volume with a charge density ρ = kr^a where k and a are constants and r is the distance from its centre. If the electric field at $\text{r} = \frac{\text{R}}{2}$ is $\frac{1}{8}$ times that at r = R, find the value of a.

Solution

From Gauss theorem,

$E \propto \frac{q}{r^2}$

(q = charge enclosed)

Therefore
$\frac{\text{E}_2}{\text{E}_1} = \frac{\text{q}_2}{\text{q}_1} = \frac{\text{r}_1^2}{\text{r}_2^2}$

or
$$8 = \frac{\int_0^{\text{R}} (4\pi\text{r}^2)\ \text{kr}^{\text{deg}}\text{dr}}{\int_0^{\text{R}/2} (4\pi\text{r}^2)\ \text{kr}^{\text{deg}}\text{dr}} \times \left(\frac{\text{R}}{2}\right)(\text{R})^2$$

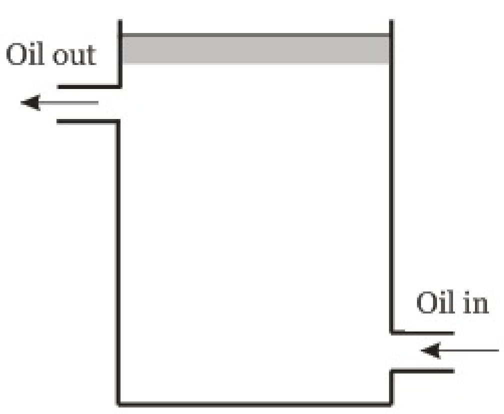
Solving this equation we get, a = 2.

24 A rod of ferromagnetic material with dimensions $20\text{cm} \times 0.5\text{cm} \times 0.1\text{cm}$ is placed in a magnetic field of strength $0.5 \times 10^4 \text{Am}^{-1}$ as a result of which a magnetic moment of 10Am^{-2} is produced in the rod. The value of magnetic induction will be ________ T

Solution

Here, $\text{V} = 20 \times 0.5 \times 0.1\text{cm}^3 = 10^{-6}\text{m}^3$
$\text{H} = 0.5 \times 10^4\text{Am}^{-1} = 5 \times 10^3\text{Am}^{-1}$
$\text{M} = 10\text{Am}^{-2}; \text{B} = ?$
$\text{I} = \frac{\text{M}}{\text{V}} = \frac{10}{10^{-6}} 10 \times 10^6\text{Am}^{-1}$
$\text{B} = \mu_0 (\text{H} + 1)$
= $4\pi \times 10^{-7} \left(5 \times 10^3 + 10 \times 10^6\right)$
$\therefore \text{B} = 4\pi \times 10^{-7} \times 5 \times 10^6 \left(10^{-3} + 2\right) = 12.6\text{T}$

25 The top of an insulated cylindrical container is covered by a disc having emissivity 0.6 and conductivity 0.167 W/Km and thickness 1 cm. The temperature is maintained by circulating oil as shown : (a) Find the radiation loss to the surrounding in W/m² if temperature of the upper surface of disc is 127°C and temperature of surrounding is 27° C. (b) Also find the temperature of the circulating oil. Neglect the heat loss due to convection.



(Given : $\sigma = \frac{17}{3} \times 10^{-8} \mathrm{Wm}^{-2} \mathrm{K}^{-4}$)

Solution

Rate of heat loss per unit area due to radiation,

$$\mathrm{I} = \mathrm{e}\sigma\left(\mathrm{T}^4 - \mathrm{T}_0^4\right)$$

Here, $\mathrm{T} = 127 + 273 = 400\mathrm{K}$

and $\mathrm{T}_0 = 27 + 273 = 300\mathrm{K}$

$$\mathrm{I} = 0 \cdot 6 \times \frac{17}{3} \times 10^{-8}\left[(400)^4 - (300)^4\right]$$

$$= 595\ \mathrm{W/m}^2$$

(b) Let θ be the temperature of the oil. The, rate of heat flow through conduction = rate of heat loss due to radiation

$$\therefore \quad \frac{\text{Temperature difference}}{\text{Thermal resistance}} = (595)\mathrm{A}$$

$$\therefore \quad \frac{(\theta - 127)}{\left(\frac{\ell}{\mathrm{KA}}\right)} = (595)\mathrm{A}$$

Here, A = area of disc ; K = thermal conductivity and ℓ =thickness (or length) of disc

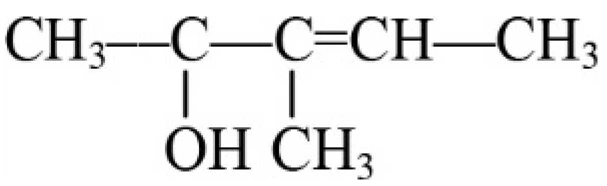
$$\therefore \quad (\theta - 127)\frac{\mathrm{K}}{\ell} = 595$$

$$\therefore \quad \theta = 595\left(\frac{\ell}{\mathrm{K}}\right) + 127$$

$$= \frac{595 \times 10^{-2}}{0 \cdot 167} + 127 = 162\,^{\circ}\mathrm{C}$$

CHEMISTRY

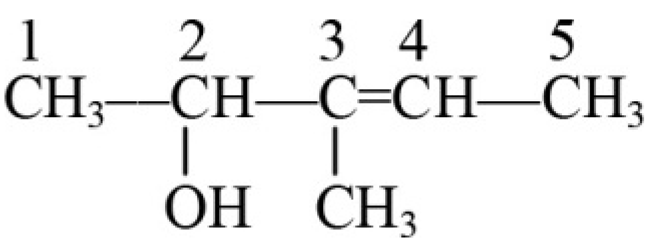
1 The IUPAC name of the given compound is:



is-

Solution

According to the priority order of functional groups, OH is given preference above the double bond. Hence, the numbering starts from the left end.



2 The presence of NH_4Cl in the test solution while precipitating group III-A hydroxides (in qualitative inorganic analysis) helps in

Solution

$$NH_4OH \rightleftharpoons NH_4^+ + OH^-$$

$$NH_4Cl \rightarrow NH_4^+ + Cl^-$$

NH_4Cl decreases the dissociation of NH_4OH by common ion effect

3 The volume strength of 1.5 N H_2O_2 is

Solution

Half Reaction of $\mathrm{H_2O_2}$ in Acidic Medium

$\mathrm{H_2O_2 + 2H^+ + 2e^- \rightarrow 2H_2O}$

$\mathrm{Eq.\ Mass} = \frac{\mathrm{Molar\ Mass}}{2}$

$\therefore\ \frac{N}{2} = M \Rightarrow$ hence 1.5 N solution $\equiv 0.75$ M solution for 1 M solution of $\mathrm{H_2O_2}$

$\mathrm{H_2O_2 \rightarrow H_2O + \frac{1}{2}O_2}$

(1 M) 1000 mL $\mathrm{H_2O_2}$ liberates 11.2 L $\mathrm{O_2}$ at STP

i.e. 1 M solution $\equiv$ 11.2 L $\mathrm{O_2}$ at NTP

0.75 M solution $\equiv (11.2)(0.75)$

= 8.4 volumes

4 The vapour pressure of water at $20^\circ\mathrm{C}$ is 17.5 mmHg.

If 18 g of glucose $(\mathrm{C_6H_{12}O_6})$ is added to 178.2 g of water at $20^\circ\mathrm{C}$, the vapour pressure of the resulting solution will be

Solution

Moles of glucose $= \frac{18}{180} = 0.1$

Moles of $\mathrm{H_2O} = \frac{178.2}{18}\ 9.9$

According to Raoult's law

$\frac{\mathrm{P^\circ - P_s}}{\mathrm{P^\circ}} = \mathrm{X_{solute}}$

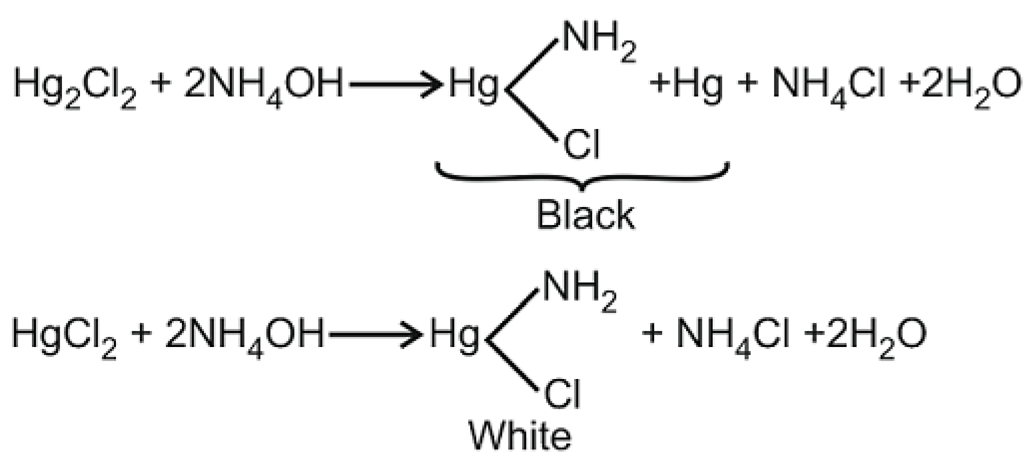
$\frac{17.5 - \mathrm{P_S}}{17.5} = \frac{0.1}{10}$

so, $\mathrm{P_s} = 17.325\ \mathrm{mm\ Hg}$

5 A metal gives two chlorides 'A' and 'B' . 'A' gives black precipitate with NH_4OH and 'B' gives white. With KI 'B' gives a red precipitate soluble in excess of KI. 'A' and 'B' are respectively :

Solution

'A' is Hg_2Cl_2 and 'B' is $HgCl_2$.



$$HgCl_2 + 2KI \rightarrow \underset{\text{Red}}{HgI_2} + 2KCl$$

$$HgI_2 + 2KI \rightarrow \underset{\text{Soluble}}{K_2HgI_4}$$

6 An acid type indicator, HIn differs in colour from its conjugate base (In^-). The human eye is sensitive to colour differences only when the ratio [In^-] / [HIn] is greater than 10 or smaller than 0.1. What should be the minimum change in the pH of the solution to observe a complete colour change $\left(K_{In} = 10 \times 10^{-5}\right)$?

Solution

The conditions when colour change of indicator will be visible are derived from equation

$$pH = pKI_{In} + \log\frac{[In^-]}{[HIn]}$$

$$pH = pKI_{In} + \log 10 = pKI_n + 1$$

$$pH = pKI_{In} + \log\frac{1}{10} = pKI_n - 1$$

$$pH_1 - pH_2 = 2$$

7 Assertion (A) : Noble gases have very low boiling points.

Reason (R) : All noble gases have general electronic configuration of $ns^2\ np^6$ (except He)

Solution

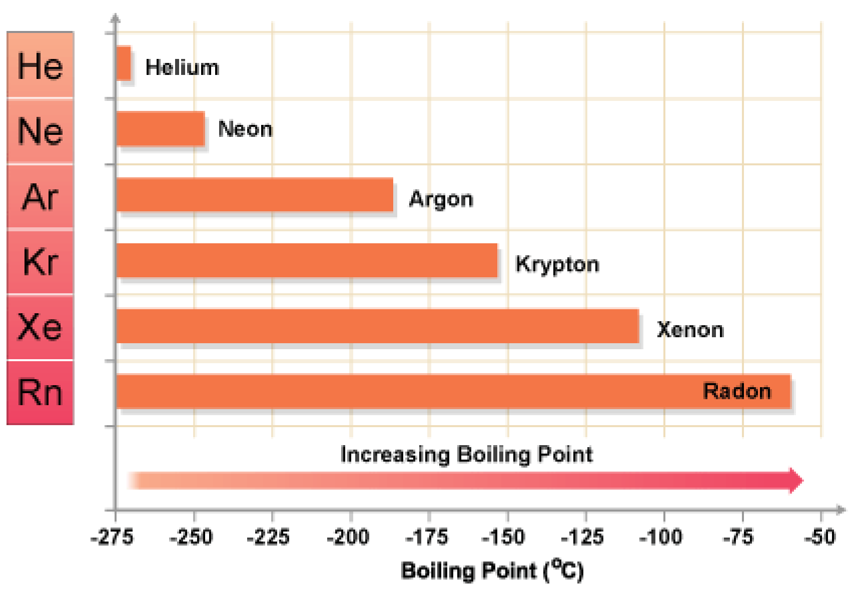
Both (A) and (R) are true but (R) is not the correct explanation of (A)

The melting and boiling points of noble gases are very low in comparison to those of other substances of comparable atomic and molecular masses. This indicates that only weak van der Waals forces or weak London dispersion forces are present between the atoms of the noble gases in the liquid or the solid state.

The van der Waals force increases with the increase in the size of the atom, and therefore, in general, the boiling and melting points increase from He to Rn.

Helium boils at −269 °C. Argon has larger mass than helium and have larger dispersion forces. Because of larger size the outer electrons are less tightly held in the larger atoms so that instantaneous dipoles are more easily induced resulting in greater interaction between argon atoms. Therefore, its boiling point (−186 °C) is more than that of He.

Similarly, because of increased dispersion forces, the boiling and melting points of monoatomic noble gases increase from helium to radon.



8 Given the molecular formula of the hexacoordinate complexes

(A) $CoCl_3 \cdot 6\ NH_3$ (B) $CoCl_3 \cdot 5\ NH_3$ (C) $CoCl_3 \cdot 4\ NH_3$

If the number of coordinated NH_3 molecules in A, B and C respectively are 6, 5 and 4, the primary valency in (A), (B) and (C) are -

Solution

Co has six coordination number, hence formulae for compounds A, B and C should be
(A) $[Co(NH_3)_6](Cl)_3$
(B) $[Co(NH_3)_5Cl](Cl)_2(NH_3)$
(C) $[Co(NH_3)_4Cl_2](Cl)(NH_3)_2$
Primary valency is equal to the oxidation state of the central metal ion.

9 For a real gas (M = 30), if density at the critical point is 0.40 g/cc and its critical temperature is $T_c = \frac{2\times10^5}{821}$K. The value of van der Waal's constant "a" in
atm L^2 mol^{-1} is

Solution

Critical volume $V_c = \frac{30 \text{ g mole}^{-1}}{0.40 \text{ g cm}^{-3}}$

$= 75 \text{ cm}^3 \text{ mole}^{-1}$

$= 0.075 \text{ L mole}^{-1}$

$V_c = 3b = 0.075 \therefore b = 0.025 \text{ L mole}^{-1}$

$T_c = \frac{8a}{27Rb} = \frac{8a}{27\times0.0821\times0.025} = \frac{2\times10^5}{821}$

$\Rightarrow \frac{8a}{27\times821\times10^{-4}\times25\times10^{-3}} = \frac{2\times10^5}{821}$

$\Rightarrow \frac{8a}{27\times25} \times 10^7 = 2 \times 10^5$

$\Rightarrow a = \frac{2\times25\times27}{8} \times 10^{-2}$

= 1.6875 atm L^2 $mole^{-1}$

10 Drained sewage has B.O.D.

Solution

Drained Sewage has more B.O.D. as it has more micro organisms.

11 Two liquids A and B are made of same elements and are diamagnetic. Liquid A on treatment with KI and starch gives blue coloured solution, however liquid B is neutral to litmus and does not give any response to starch iodide paper, then A and B will be

Solution

$H_2O_2 + KI + \text{starch} \rightarrow$Blue colour

Liberated I_2 makes starch paper blue while H_2O will not.

12 The equilibrium constant for the reaction,

$N_2(g) + O_2(g) \rightleftharpoons 2NO(g)$

at temperature T is 4×10^{-4}. The value of K_c for the reaction

$NO(g) \rightleftharpoons \frac{1}{2}N_2(g) + \frac{1}{2}O_2(g)$

at the same temperature is

Solution

$N_2\,(g) + O_2(g) \rightleftharpoons 2NO(g)$

$K_c = \frac{[NO]^2}{[N_2][O_2]} = 4 \times 10^{-4}$

$NO \rightleftharpoons \frac{1}{2}N_2\,(g) + \frac{1}{2}O_2\left(g\right)$

$K_c^{'} = \frac{[N_2]^{1/2}[O_2]^{1/2}}{[NO]}$

$= \sqrt{\frac{1}{K_c}} = \sqrt{\frac{1}{4\times10^{-4}}} = 50$

13 Which of the following statements is correct?

Solution

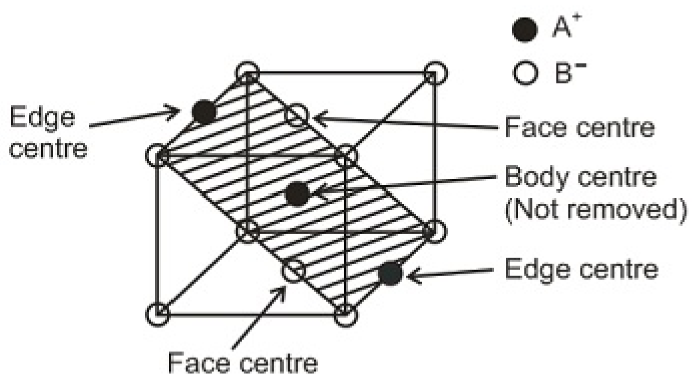
Because of the greater difference in electronegativity between C and O, the energy of the molecular orbitals in CO are in the following order

$\sigma_{1s}^2 \sigma_{1s}^{*2}\, \sigma_{2s}^2\, \left[\pi_{2p_y}^2 = \pi_{2p_z}^2\right] \sigma_{2p_z}^2\, \sigma_{2s}^{*2}$

Bond order in CO $= \frac{1}{2}\ (10 - 4) = 3$

14 In a solid having rock salt structure, if all the atoms touching one body diagonal plane are removed (except at body centre), then the formula of the left unit cell is

Solution



• 4 $B^{\ominus}$ from corner and 2 $B^{\ominus}$ from face center are removed

• 2 $A^{\oplus}$ from edge center are removed (body centered is not removed)

NaCl (rock salt)-type structure is fcc type.

$(\therefore\ \ Z_{eff} = 4) \qquad \left[Z_{eff}\left(Na^{\oplus}\right) = 4, Z_{eff}\left(Cl^{\ominus}\right) = 4\right]$

In NaCl structure anion ($Cl^{\ominus}$) (here $B^{\ominus}$) are present at corners and face center while cations ($Na^{\oplus}$) (here $A^{\oplus}$) are present in all OVs (present at body center and edge center).

∴ Number of $B^{\ominus}$ ions removed

$= 4 \times \frac{1}{8}\ \ (\text{corner share}) + 2 \times \frac{1}{2}\ \ (\text{face center share})$

$= \frac{1}{2} + 1 = 3/2$

Number of $A^{\oplus}$ ions removed $= 2 \times \frac{1}{4}\ \ (\text{edge center share})$

$= \frac{1}{2}$

No. of A⁻ ions left $= 4 - \frac{1}{2} = 3.5$

No. of B⁺ ions left $= 4 - 1.5 = 2.5$

Formula of balance compound $A_{3.5}B_{2.5}$

15 On passing 0.1 F of electricity through molten solution of. Al_2O_3, amount of aluminium metal deposited at cathode is (Al = 27)

Solution

At cathode,

$\mathrm{Al}^{3+} + 3e^{-} \rightarrow \mathrm{Al}$

$E_{\mathrm{Al}} = \frac{27}{3} = 9$

$w_{\mathrm{Al}} = E_{\mathrm{Al}} \times$ no.of faradays

$= 9 \times 0.1 = 0.9\ \mathrm{g}$

16 What is the atomic number of the element with symbol Uus?

Solution

Atomic number of the element with symbol Uus= 117

where Uus = ununseptium or tennessine is the superheavy (very heavy), man made chemical element.

17 n and ℓ for four electrons are given as $\mathrm{(i)\,n} = 4,\ \ell = 1\ \mathrm{(ii)\,n} = 4,\ \ell = 0\ \mathrm{(iii)\,n} = 3,\ \ell = 2\ \mathrm{(iv)\,n} = 3,\ \ell = 1$. Arrange them in order of increasing energy from the lowest to highest:

Solution

Lower the value of $(\mathrm{n} + \ell)$, lower is the energy. When $(\mathrm{n} + \ell)$ value is the same for two orbitals, lower n value implies lower energy of the electron. Thus, the order will be:
(iv) < (ii) < (iii) < (i).

18 The end product of (4n + 2) disintegration series is:

Solution

The $4n + 2$ series ends at an isotope of lead which has a mass number of the type $4n + 2$ where 'n' is an integer

19 Consider the reaction $\mathrm{A} \rightarrow 2\,\mathrm{B} + \mathrm{C},\ \Delta\mathrm{H} = -15\ \mathrm{kcal}$. The energy of activation of backward reaction is 20 kcal mol⁻¹. In presence of catalyst, the energy of activation of forward reaction is 3 kcal mol⁻¹. At 400 K the catalyst causes the rate of the forward reaction to increase by the number of times equal to-

Solution

$\mathrm{E_{a(f)}} - \mathrm{E_{a(b)}} = \Delta\mathrm{H} = -15\ \mathrm{kcal}$

$\Rightarrow \quad \mathrm{E_{a(f)}} = -15 + 20 = 5\ \mathrm{kcal}$

$\frac{\mathrm{k_{(catalyst)}}}{\mathrm{k_a}} = \mathrm{e}^{\mathrm{E_a} - \mathrm{E_{a(catalyst)}}/\mathrm{RT}}$

$= \mathrm{e}^{(5-3) \times 10^3 / 2 \times 400}$

$= \mathrm{e}^{2.5}$

20 A 20 cm^3 mixture of CO, CH_4 and He gases was exploded by an electric discharge at room temperature with excess oxygen. The volume contraction was found to be 13.0 cm^3 . A further contraction of 14.0 cm^3 occurred when the residual gas was treated with KOH solution. Find out the composition of CH_4 in the mixture in terms of volume percentage.

Solution

Let a and b cm^3 be the volumes of CO and CH_4 respectively in the mixture.

$$\begin{array}{ccccc} CO_{(g)} & + & \frac{1}{2}O_{2(g)} & \rightarrow & CO_{2(g)} \\ 1 \text{ vol} & & 1/2 \text{ vol} & & 1 \text{ vol} \\ a \text{ cm}^3 & & a/2 \text{ cm}^3 & & a \text{ cm}^3 \end{array}$$

$$\begin{array}{ccccccc} CH_{4(g)} & + & 2O_{2(g)} & \rightarrow & CO_{2(g)} & + & 2H_2O_{(\ell)} \\ 1 \text{ vol} & & 2 \text{ vol} & & 1 \text{ vol} & & \\ b \text{ cm}^3 & & 2b \text{ cm}^3 & & b \text{ cm}^3 & & \end{array}$$

Reactions show that volume concentration after the reaction is due only to the consumption of oxygen.

Volume contraction after reaction, $\frac{a}{2} + 2b = 13$ ...... (i)

When treated with aqueous KOH, CO_2 is absorbed.

Hence, $a + b = 14$ ...... (ii)

Solving equation (i) and (ii), a = 10, b = 4

% of CO = 50 ; **% of CH_4 = 20** ; % of He = 30 ;

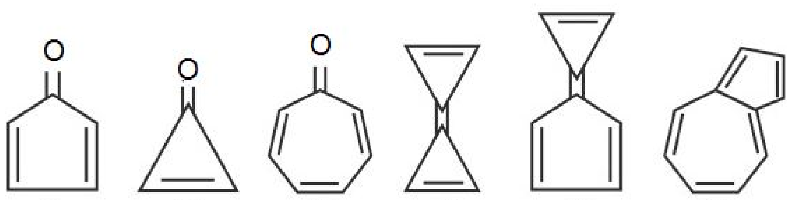
21 How many of the following species are related to Hall's process of purification of bauxite?
White bauxite, Na_2CO_3, CO_2, cryolite, red bauxite, NaOH

Solution

Na_2CO_3, CO_2 and red bauxite are related to Hall's process of purification of bauxite.

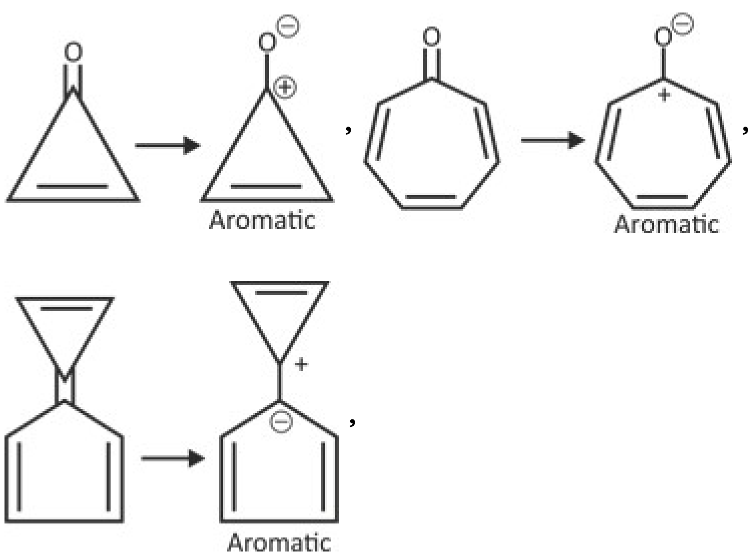
$2\ Al_2O_3 + 3C \rightarrow 4\ Al + 3CO_2$

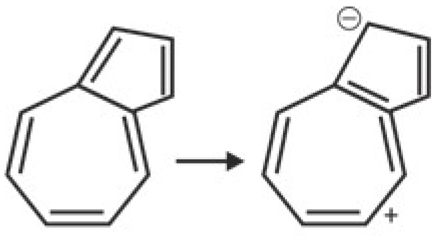
22 Find out total number of compounds which are more stable in its ionic form



(1) (2) (3) (4) (5) (6)

Solution

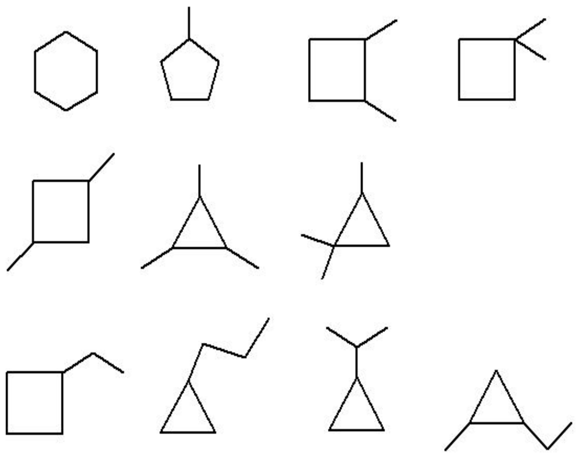


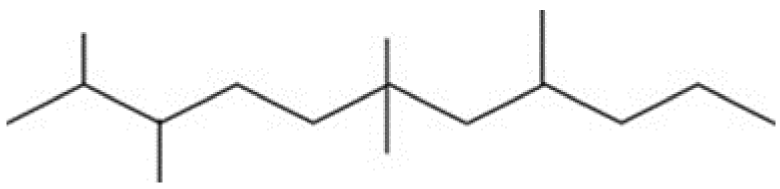


23 The total number of cyclic isomers of formula C_6H_{12} is/are

Solution

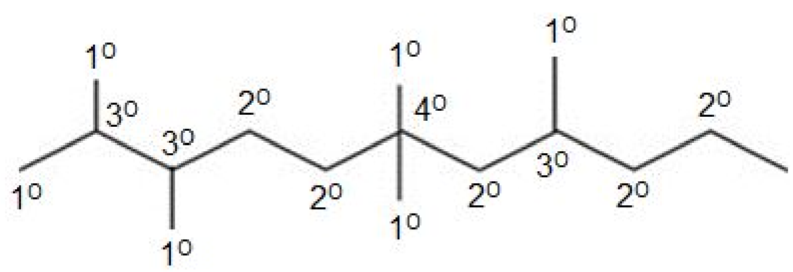
Total 11 cyclic isomer are shown below here.



24 

In this molecule
$X =$ no . of $1^o -$ carbon atoms
$Y =$ no . of $2^o -$ carbon atoms
$Z =$ no . of $3^o -$ carbon atoms
$M =$ no . of 4^o $-$ carbon atoms
What is the value of $\frac{X+Y+Z+M}{2}$?

Solution



$X = 7$
$Y = 5$
$Z = 3$
$M = 1$
$= \frac{X+Y+Z+M}{2}$
$= \frac{16}{2}$
$= 8$

25 For the reaction cis-2-butene $\longrightarrow$ trans-2-butene and cis-2-butene $\longrightarrow$ 1-butene, $\Delta H^\circ = -960$ and +1771 cal/mol, respectively. The heat of combustion of 1- butene is -649.8 kcal/mol. Determine the heat of combustion of trans -2- butene. Also calculate the bond energy of C = C bond in trans - 2 - butene. Given BE of C = O = 196, O - H = 110, O = O = 118, C - C = 80 and C - H = 98 kcal/mol, respectively.

Solution

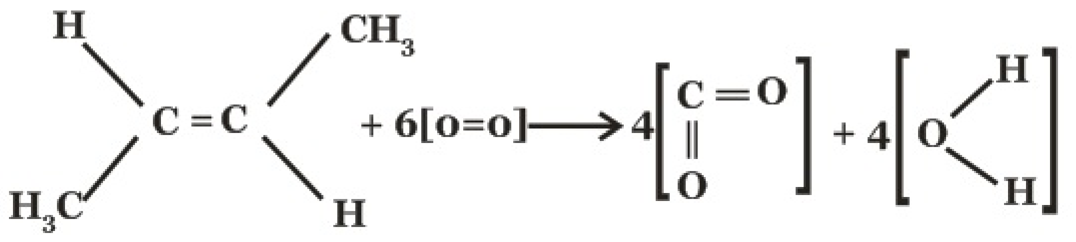
$$\text{cis-2-Butene} \dashrightarrow \text{trans-2-Butene} \quad \Delta H^\circ = -0 \cdot 96 \text{ kcal}$$
$$\text{cis-2-Butene} \dashrightarrow \text{1-Butene} \quad \Delta H^\circ + 1 \cdot 771 \text{ kcal}$$

Subtracting

1Butene $\longrightarrow$ trans-2-Butene $\quad \Delta H^\circ = -0 \cdot 96 - 1.771 = -2.731 \text{Kcal}$

1Butene$C_4H_{8(g)} + 6O_{2(g)} \longrightarrow 4Co_{2(g)} + 4H_2O\,(l)\Delta H^\circ = -649 \cdot 8$ kcal

trans-2-Butene $\longrightarrow$ 1-Butene $\Delta H^\circ + 2.731$ kcal

Adding $\Rightarrow$ 

$\Delta H^0 = -649.8 + 2.73 = -647.07 \text{Kcal}$

$-647.07 = [BE_{C=C} + 2BE_{C-H} + 2BE_{C-C} + 6BE_{C-H}] + 6\,[BE_{O=O}] - 8BE_{C=0} - 8BE_{O-H}$

$[BE_{C=C} + 8 \times 98 + 2 \times 80] + 6\,[118] - 8\,[196] - 8\,[110] = -647 \cdot 07$

$BE_{C=C} + 784 + 160 + 708 - 1568 - 880 = -647 \cdot 07$

$BE_{C=C} = -647 + 2448 - 1652 = 149$ kcal/mole

MATHEMATICS

1 The system of equations $x + 2y + 3z = 1,\ 2x + y + 3z = 2$ and $5x + 5y + 9z = 5$ has

Solution

Given system of equations is $x + 2y + 3z = 1,\ \ 2x + y + 3z = 2$ and $5x + 5y + 9z = 5$

Now, $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 5 & 5 & 9 \end{vmatrix}$

$= 1\,(9 - 15) - 2\,(18 - 15) + 3\,(10 - 5)$

$= -6 - 6 + 15$

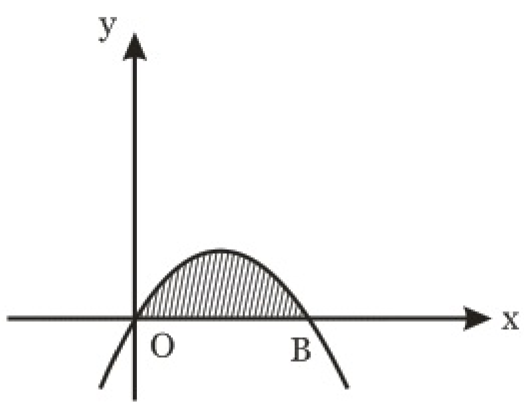
$= 3 \neq 0$

Hence, it has unique solution

2 The values of m for which, area of the region bounded by the curve $y = x - x^2$ and the line y = mx is equal to $^9/_2$ sq. unit, are

Solution

Case I : When m = 0
In this case $\quad$ y = x - x² $\quad$ ...(i)
and $\quad$ y = 0 $\quad$ ...(ii)
are two given curves, y>0 is total region above x-axis.
Therefore, area between y = x - x² and y = 0
is area between y = x - x² and above the x-axis



$$\therefore A = \int_0^1 \left(x - x^2\right)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

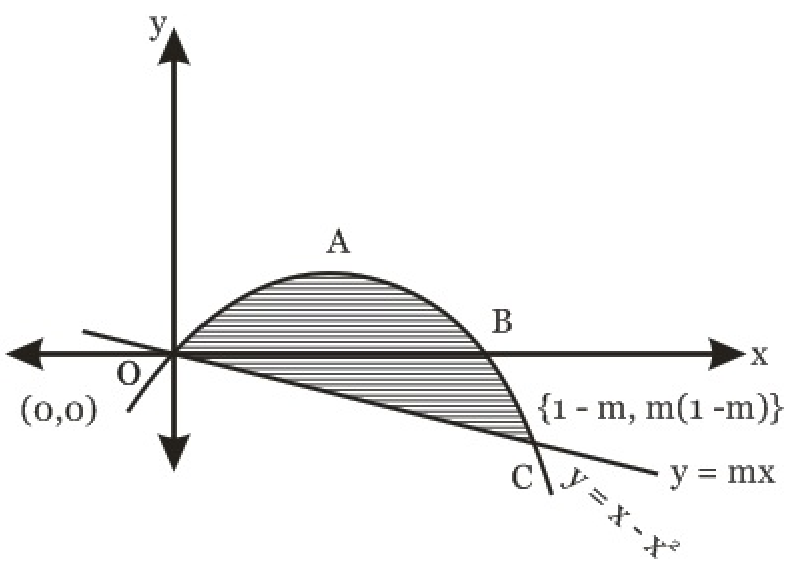
$$= \frac{1}{6} \neq \frac{9}{2}$$

Hence, m cannot be zero.

Case II : When $m < 0$
in this case area between
$y = x - x^2$ and $y = mx$ is

OABCO and points of intersection are (0,0) and
$\{1 - m, m(1 - m)\}$

Area of curve OABCO $= \int_0^{1-m} \left[x - x^2 - mx\right]dx$



$$= \left[(1 - m)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1-m}$$

$$= \frac{1}{2}(1 - m)^3 - \frac{1}{3}(1 - m)^3 = \frac{1}{6}(1 - m)^3$$

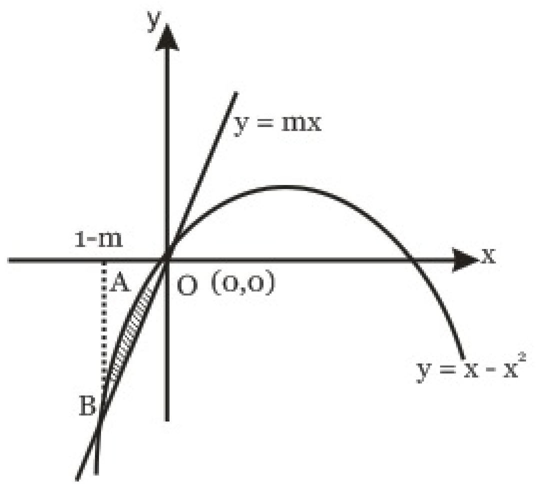
$\therefore \frac{1}{6}(1 - m)^3 = \frac{9}{2}$ (given)

$$\Rightarrow (1 - m)^3 = 27$$

$$\Rightarrow 1 - m = 3$$

$$\Rightarrow m = -2$$

Case III : When $m > 0$
In this case, $y = mx$ and $y = x - x^2$ intersect in (0,0) and $\{(1 - m), m(1 - m)\}$ as shown in Fig.



Area of shaded region $= \int_{1-m}^{0} (x - x^2 - mx)dx$

$= \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_{1-m}^{0}$

$= -\frac{1}{2}(1-m)(1-m)^2 + \frac{1}{3}(1-m)^3$

$= -\frac{1}{6}(1-m)^3$

$\Rightarrow \frac{9}{2} = -\frac{1}{6}(1-m)^3$ (given)

$\Rightarrow (1-m)^3 = -27$

$\Rightarrow (1-m) = -3$

$\Rightarrow m = 3 + 1 = 4$

Therefore, (-2) and (4) are the answers.

3 If $\alpha,\ \beta$ are the roots of the equation $6x^2 - 5x + 1 = 0$, then the value of tan($\tan^{-1}\alpha + \tan^{-1}\beta$) is

Solution

As $\alpha + \beta = \frac{5}{6}\ \alpha\beta = \frac{1}{6}$

So, $\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\frac{\alpha+\beta}{1-\alpha\beta} = \tan^{-1}\left(\frac{\frac{5}{6}}{1-\frac{1}{6}}\right)$

4 Let $\alpha, \beta\ (\alpha < \beta)$ are roots of the equation $x^2 + bx + c = 0$ where $c < 0 < b,$ then

Solution

We have $\alpha + \beta = -b\ ,\ \ \alpha\beta = c$

As $c < 0, b > 0$ we get $\alpha < 0 < \beta$

Also, $\beta = -b - \alpha < \ -\alpha = |\alpha|$

Thus $\alpha < 0 < \beta < |\alpha|$

5 If $A\ =\ \{2,\ 3,\ 5\},\ B\ = \{2,\ 5,\ 6\}$, then $(A - B) \times\ (A \cap B)$ is

Solution

Given $A\ =\ \{2,\ 3,\ 5\},\ B\ = \{2,\ 5,\ 6\}$

Now $A - B = \{3\}$ and $A \cap B = \{2{,}5\}$

Therefore $(A - B) \times (A \cap B) = \{(3,\ 2);\ (3,\ 5)\}$

Hence $\{(3,\ 2),\ \ (3,\ 5)\}$ is the correct answer.

6 Let P(n) denote the statement that n^2+n is odd. It is seen that $P(n) \Rightarrow P(n+1),$ then P_n is true for all

Solution

$\mathrm{n}^2+\mathrm{n}=\mathrm{n}(\mathrm{n}+1),$ which is product of 2 consecutive natural numbers is always even.

7 The solution of the differential equation $\sec^2 x \tan\ \mathrm{y}\ dx+\sec^2 \mathrm{y}\ \ \tan\ \mathrm{x}\ dy=0$ is { Where C is an arbitrary constant }

Solution

Given, $\frac{\sec^2 x}{\tan x} dx=-\frac{\sec^2 y}{\tan y} dy$

$\Rightarrow \quad \int \frac{\sec^2 x}{\tan x} dx=-\int \frac{\sec^2 y}{\tan y} dy$

Put $\tan\ x=u$

$\Rightarrow \quad \sec^2 x\ dx=du$

And $\tan\ \mathrm{y}=v$

$\Rightarrow \quad \sec^2 y\ dy=dv$

$\therefore \quad \int \frac{du}{u}=-\int \frac{dv}{v}$

$\Rightarrow \quad \log u=-\log v+\log c \Rightarrow uv=c$

$\therefore \quad \tan\ x.\tan\ y=c$

8 $\frac{d}{dx}\left\{\begin{array}{c}\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) \\ -\tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)\end{array}\right\}$ is equal to $\left\{|x|<\sqrt{2}-1\right\}$

Solution

Let $I=\frac{d}{dx}\left\{\begin{array}{c}\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) \\ -\tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)\end{array}\right\}$

Put $x=\tan\theta$ the given equation

$\therefore \quad I=\frac{d}{dx}\left\{\tan^{-1}(\tan 2\theta)+\tan^{-1}(\tan 3\theta)-\tan^{-1}(\tan 4\theta)\right\}$

$=\frac{d}{dx}(\theta)=\frac{d}{dx}\left(\tan^{-1} x\right)=\frac{1}{1+x^2}$

9 If the length of subnormal is equal to length of sub-tangent at point (3, 4) on the curve y = f(x) and the tangent at (3, 4) to y = f(x) meets the coordinate axes at A and B, then maximum area of the ΔOAB where O is origin, is

Solution

Length of subnormal = length of subtangent

$\Rightarrow \left|\mathrm{y}_1\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)_{(\mathrm{x}_1\mathrm{y}_1)}\right|=\left|\frac{\mathrm{y}_1}{\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)_{(\mathrm{x}_1\mathrm{y}_1)}}\right|$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1y_1)} = \pm 1$

If $\left(\frac{dy}{dx}\right)_{(x_1y_1)} = 1$

Then the equation of tangent y - x = 1

and area of ΔOAB $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

If $\left(\frac{dy}{dx}\right)_{(x_1y_1)} = -1$

Then the equation of tangent is x + y = 7

and area of ΔOAB $= \frac{1}{2} \times 7 \times 7 = \frac{49}{2}$

10 Suppose α, β, γ and δ are the interior angles of regular pentagon, hexagon, decagon and dodecagon respectively, then the value of $|\cos\alpha \sec\beta \cos\ \gamma\ \mathrm{cosec}\,\delta|$ is_______.

Solution

Interior angle of regular polygon of side n is $\left(180^\circ - \frac{360^\circ}{n}\right)$

Hence, $\alpha = 108^\circ; \beta = 120^\circ; \gamma = 144^\circ; \delta = 150^\circ$

$\cos\alpha = \cos 108^\circ = -\sin 18^\circ = -\left(\frac{\sqrt{5}-1}{4}\right)$

$\sec\beta = \sec 120^\circ = -2$

$\cos\gamma = \cos 144^\circ = -\cos 36^\circ = -\left(\frac{\sqrt{5}+1}{4}\right)$

$\mathrm{cosec}\,\delta = \mathrm{cosec}\,150^\circ = +2$

$\therefore \quad \left|\left(\frac{\sqrt{5}-1}{4}\right)(2)\left(\frac{\sqrt{5}+1}{4}\right)(-2)\right| = 1$

11 The value of $\sum_{r=1}^{18} \cos^2(5r)^o$, where x^o denotes the x degrees, is equal to

Solution

$\sum_{r=1}^{18} \cos^2(5r)^o = \cos^2 5^o + \cos^2 10^o + \cos^2 15^o + \ldots + \cos^2 85^o + \cos^2 90^o$

$= (\cos^2 5^\circ + \cos^2 85^\circ) + (\cos^2 10^\circ + \cos^2 80^\circ) + (\cos^2 15^\circ + \cos^2 75^\circ) + ... + (\cos^2 40^\circ + \cos^2 50^\circ) + \cos^2 45^\circ$

$= (\cos^2 5^o + \sin^2 5^o) + (\cos^2 10^o + \sin^2 10^o) + (\cos^2 15^o + \sin^2 15^o) + \ldots + (\cos^2 40^o + \sin^2 40^o) + \cos^2 45^o$

$=1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \frac{1}{12} = 8 + \frac{1}{2} = \frac{17}{2}$

12 There are "m" bags which are numbered by "m" consecutive positive integers starting with the number k. Each bag contains as many different flowers as the number labeled against the bag. A boy has to pick up k flowers from any one of the bags. In how many different ways he can do the work?

Solution

First bag contains " k " flowers 2nd bag contains " $k+1$ " flowers, similarly m^{th} bag contains $k+m-1$ flowers, selecting " k " flowers from any bag

now the total number of ways

$= {}^{k}C_k + {}^{k+1}C_k + {}^{k+2}C_k + ... + {}^{k+m-1}C_k$

$= {}^{k+1}C_{k+1} + {}^{k+1}C_k + {}^{k+2}C_k \ldots\ldots {}^{k+m-1}C_k \ \left[\because {}^{k}C_k = {}^{k+1}C_{k+1} = 1\right]$
$= {}^{k+2}C_{k+1} + {}^{k+2}C_k + {}^{k+3}C_k + \ldots\ldots + {}^{k+m-1}C_k \left[\because {}^{n}C_k + {}^{n}C_{k+1} = {}^{n+1}C_{k+1}\right]$ put it in side colume
$= {}^{k+3}C_{k+1} + {}^{k+3}C_k + \ldots + {}^{k+m-1}C_k$
$= {}^{k+m-1}C_{k+1} + {}^{k+m-1}C_k$
$= {}^{k+m}C_k$

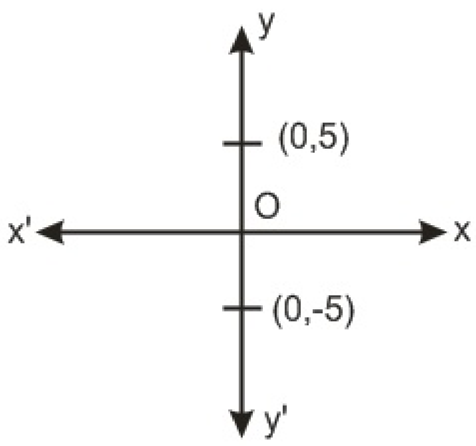
13 The complex numbers z = x + iy which satisfy the equation $\frac{|\mathrm{z}-5\mathrm{i}|}{|\mathrm{z}+5\mathrm{i}|} = 1$, lie on

Solution

Given, $|\frac{z-5i}{z+5i}|$= 1

$\Rightarrow |\mathrm{z}-5\mathrm{i}| = |\mathrm{z}+5\mathrm{i}|$

(if $|\mathrm{z}-\mathrm{z}_1| = |\mathrm{z}-\mathrm{z}_2|$, then it is a perpendicular bisector of z_1 and z_2)



and Perpendicular bisector of (0, 5) and (0, - 5) is x-axis,

Therefore z will lie on x - axis

14 $\int \tan\left(\sin^{-1} x\right) dx$ is equal to

Solution

Let, $I = \int \tan(\sin^{-1} x)\, dx$

$= \int \tan\left(\tan^{-1} \frac{x}{\sqrt{1-x^2}}\right) dx = \int \frac{x}{\sqrt{1-x^2}} dx$

Put, $1 - x^2 = t^2$

$\Rightarrow \quad -2x\, dx = 2tdt$

$\therefore \quad I = -\int \frac{tdt}{t} = -t + c = -\sqrt{1-x^2} + c$

15 The value of $\int\limits_0^{\pi} |\sin^3\theta| d\theta$ is

Solution

Let $I = \int\limits_0^{\pi} \sin^3\theta d\theta$

$\left[\begin{array}{c} \because sin\theta > 0 \\ for\ 0 < \theta < \pi \end{array}\right]$

$= \int_0^{\pi} \sin\theta(1-\cos^2\theta)d\theta$

Put $\cos\theta = t \Rightarrow \; -\sin\theta d\theta = dt$

$\therefore I = \int_{-1}^{1}\left(1-t^2\right)dt = \left[t-\frac{t^3}{3}\right]_{-1}^{1}$

$= 2-\frac{2}{3}$

$= \frac{4}{3}$

16 If f(x) is a polynomial satisfying $\mathrm{f}(\mathrm{x})\mathrm{f}(1/\mathrm{x}) = \mathrm{f}(\mathrm{x}) + \mathrm{f}(1/\mathrm{x})$ and $\mathrm{f}(2) > 1$, then $\lim\limits_{\mathrm{x}\to 1}\mathrm{f}(\mathrm{x})$ is :

Solution

Method I:

Given, $\mathrm{f}(\mathrm{x}).\mathrm{f}(1/\mathrm{x}) = \mathrm{f}(\mathrm{x}) + \mathrm{f}(1/\mathrm{x}), \mathrm{f}(2) > 1$

let $\mathrm{f}(\mathrm{x}) = \mathrm{a}_0\mathrm{x}^{\mathrm{n}} + \mathrm{a}_1\mathrm{x}^{\mathrm{n}-1} + \ldots + \mathrm{a}_{\mathrm{n}-1}\mathrm{x} + \mathrm{a}_{\mathrm{n}},\ \mathrm{a}_0 \neq 0$

Then by comparing the coefficients of like powers, we get

$\mathrm{a}_{\mathrm{n}} = 1, \mathrm{a}_0^2 = 1, \mathrm{a}_1 = \mathrm{a}_2 = \ldots = \mathrm{a}_{\mathrm{n}-1} = 0$

$\therefore \quad \mathrm{f}(\mathrm{x}) = -\mathrm{x}^{\mathrm{n}} + 1$ or $\mathrm{f}(\mathrm{x}) = \mathrm{x}^{\mathrm{n}} + 1$

Then $\quad \mathrm{f}(2) > 1 \Rightarrow \mathrm{f}(\mathrm{x}) = \mathrm{x}^{\mathrm{n}} + 1$

$\therefore \lim\limits_{\mathrm{x}\to 1}\mathrm{f}(\mathrm{x}) = \lim\limits_{\mathrm{x}\to 1}(\mathrm{x}^{\mathrm{n}} + 1) = 2$

Method II:

Given, $f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$
Let $f(x) = 1 + x^n \quad (\because\ f(2) > 1)$

then $\lim\limits_{x\to 1} f(x) = 2$

17 If there is a term containing $\mathrm{x}^{2\mathrm{r}}$ in $\left(\mathrm{x}+\frac{1}{\mathrm{x}^2}\right)^{\mathrm{n}-3}$ $(\mathrm{n} \in \mathrm{N})$ then

Solution

The given expression is $\left(\mathrm{x}+\frac{1}{\mathrm{x}^2}\right)^{\mathrm{n}-3}$, then

the general term of $\left(\mathrm{x}+\frac{1}{\mathrm{x}^2}\right)^{\mathrm{n}-3}$ is

$\mathrm{t}_{\mathrm{k}+1} = {}^{\mathrm{n}-3}\mathrm{C}_{\mathrm{k}}\mathrm{x}^{\mathrm{n}-3-\mathrm{k}}\left(\frac{1}{\mathrm{x}^2}\right)^{\mathrm{k}}$

$\Rightarrow \qquad \mathrm{t}_{\mathrm{k}+1} = {}^{\mathrm{n}-3}\mathrm{C}_{\mathrm{k}}\ \mathrm{x}^{\mathrm{n}-3(\mathrm{k}+1)}$

There is a term containing x^{2r}, if

n - 3(k + 1) = 2r

⇒ n - 2 r = 3 (k + 1), k ∈ N

∴ n - 2 r is a positive integral multiple of 3.

18 A man standing on a horizontal plane observes the angle of elevation of the top of a tower to be α . After walking a distance equal to double the height of the tower the angle of the elevation becomes 2α, then α is equal to:

Solution

19 A and B each toss three coins. The probability that both get the same number of heads, is

Solution

The probability that A get r heads in the three tosses of a coin, is

$$P(X = r) = {}^3C_r\left(\frac{1}{2}\right)^3.$$

The probability that A and B both get r heads in three tosses of a coin, is

$$ {}^3C_r\left(\frac{1}{2}\right)^3 \cdot {}^3C_r\left(\frac{1}{2}\right)^3 $$

$$= \left({}^3C_r\right)^2\left(\frac{1}{2}\right)^6$$

∴ Required probability = $\sum_{r=1}^{3}\left({}^3c_r\right)^2\left(\frac{1}{2}\right)^6$

$$= \left(\frac{1}{2}\right)^6(1+9+9+1) = \frac{20}{64} = \frac{5}{16}$$

20 If $f(x) = \begin{cases} Ax+3, & x<1 \\ 2, & x=1 \\ B+x^2 & x>1 \end{cases}$ $\text{is continuous } \forall \text{ x} \in \text{R then } (\text{A},\text{B})\text{is}-$

Solution

$$f(1^+) = B+1,\ f(1) = 2,\ f(1^-) = 3+A$$

$f(x)$ is continuous at $x=1$

$$B+1 = 2 \Rightarrow B = 1$$

$$A+3 = 2 \ \Rightarrow \ A = \ -1$$

21 Number of solutions of $2^{\sin(|x|)} = 3^{|\cos x|}$ in $[-\pi, \pi]$, is equal to

Solution

In the interval $[-\pi,\ \pi]$ all the values of $\sin|x|$ are positive as well as $|\cos x|$
Hence in $[-\pi,\ \pi]$, equation gets reduced to,
$2^{\sin x} = 3^{\cos x}$
Taking log on both sides,
$\sin x \log 2 = \cos x \log 3$
$\Rightarrow \ |\tan x| = \log 3/\log 2$, value of $\tan x$ are positive in 1^{st} and 3^{rd} quadrant and negative in 2^{nd} and 4^{th} quadrant but positive for $|\tan x|$ in all the values of x in $[-\pi,\ \pi]$.

Hence, $\tan x$ will repeat its value in all 4 quadrants, so the number of solutions are 4 .

22 Let $f(x) = \sin\frac{x}{3} + \cos\frac{3x}{10}$ for all real x. The least natural number n such that $f(10n\pi + x) = f(x)$ for all real values of x is

Solution

$f(x)=\sin\frac{x}{3}+\cos\frac{3x}{10}$

Period of $\sin\frac{x}{3}=6\pi$

Period of $\cos\frac{3x}{10}=\frac{20\pi}{3}$

$\left(\because \quad \sin a\theta, \cos a\theta \text{ are periodic with period} \frac{2\pi}{|a|}\right)$

LCM of $\left(6\pi, \frac{20\pi}{3}\right)=60\pi$

Therefore, the period of $f(x)=60\pi$

Hence, $n=6$.

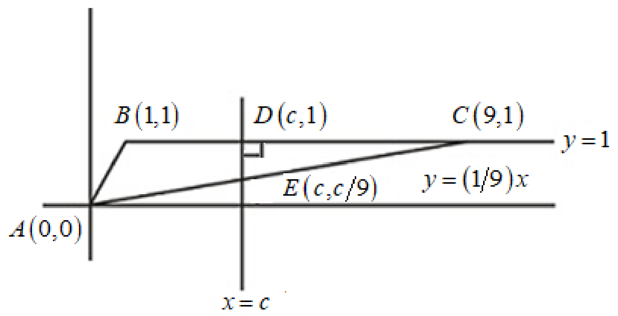
23 The line $x=c$ cuts the triangle with corners as points (0, 0), (1, 1) and (9, 1) into two regions. For the area of the two regions to be same, the value of c must be equal to:

Solution

Let $A(0,\,0)$, $B(1,\,1)$ and $C(9,\,1)$, and line $x=c$ cuts side BC at $D(c,\,1)$ and AC at $E\left(c,\,\frac{c}{9}\right)$ now area of ΔABC $\Delta=\frac{1}{2}\begin{vmatrix}0&0&1\\9&1&1\\1&1&1\end{vmatrix}$

$\Delta=4$

Now the line $x=c$ bisect the area of triangle in two equal parts so the area of the both the parts of the triangle must be two.



Area of $\Delta CDE=\frac{1}{2}\Delta ABC$

$\Rightarrow \frac{1}{2}bh=\frac{1}{2}(4)(\therefore b=DE=\left(1-\frac{c}{9}\right)\&h=CD=(9-c))$

$\Rightarrow \frac{1}{2}\times\left(1-\frac{c}{9}\right)\times(9-c)=2$

$\Rightarrow (9-c)^2=36$

$\Rightarrow 9-c=\pm6 \Rightarrow c=15$ or $c=3$

$c=3$ as $c\in(0,\,9)$

24 Define $a_k=\left(k^2+1\right)k!$ and $b_k=a_1+a_2+a_3+\ldots\ldots\ldots+a_k$. Let $\frac{a_{100}}{b_{100}}=\frac{m}{n}$ where m and n are relatively prime numbers. Then the value of $(n-m)$ is equal to _____

Solution

$a_k=\left(k^2+1\right)k!=(k(k+1)-(k-1))k!=k(k+1)!-(k-1)k!$

$a_1=1\cdot 2!-0$

$a_2=2\cdot 3!-1\cdot 2!$

$a_3=3\cdot 4!-2\cdot 3!$

⋮

$$\frac{a_k = k(k+1)! - (k-1)k!}{\text{Hence } b_k = k(k+1)!}$$

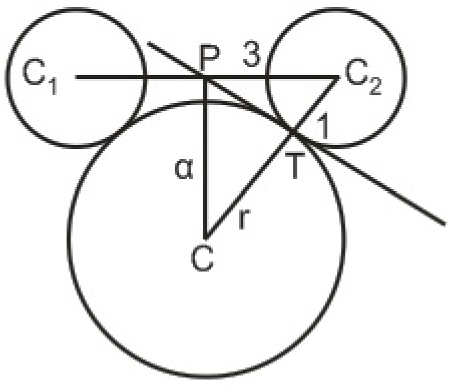
$$\therefore \frac{a_k}{b_k} = \frac{(k^2+1)k!}{k(k+1)!} = \frac{(k^2+1)}{k(k+1)} = \frac{k^2+1}{k^2+k}$$

$$\frac{a_{100}}{b_{100}} = \frac{10001}{10100} = \frac{m}{n}; (n-m) = 99 \text{ Ans.}$$

25 The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the midpoint of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius (in units) of the circle C is

Solution

In $\Delta \mathrm{CPC_2},\ (\mathrm{r}+1)^2 = \alpha^2 + 3^2 \quad \ldots$(i)



In $\Delta \mathrm{PTC}$,

$$\alpha^2 = \mathrm{r}^2 + \left(3^2 - 1^2\right)$$

$$\alpha^2 = \mathrm{r}^2 + 8$$

Putting the value of α^2 in equation (i), we get,

$$1 + \mathrm{r}^2 + 2\mathrm{r} = 1\left(\mathrm{r}^2 + 8\right) + 9$$

$$\Rightarrow 2\mathrm{r} = 16$$
$$\Rightarrow \mathrm{r} = 8$$